

Analytical study of superradiant instability for five-dimensional Kerr-Gödel black hole

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Abstract

We present an analytical study of superradiant instability of rotating asymptotically Gödel black hole (Kerr-Gödel black hole) in five-dimensional minimal supergravity theory. By employing the matched asymptotic expansion method to solve Klein-Gordon equation of scalar field perturbation, we show that the complex parts of quasinormal frequencies are positive in the regime of superradiance. This implies the growing instability of superradiant modes. The reason for this kind of instability is the Dirichlet boundary condition at asymptotic infinity, which is similar to that of rotating black holes in anti-de Sitter (AdS) spacetime.

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I. INTRODUCTION

Superradiance is a classical phenomenon associated with ergosphere in rotating black hole [1–4]. If the modes of impinging bosonic fields $\Phi \sim \exp\{-i\omega t + im\phi\}$, with frequency ω and angular momentum m , are scattered off by event horizon of black hole, the requirement of superradiant amplification is $0 < \omega < m\Omega_H$, where Ω_H is angular velocity of event horizon.

The superradiance phenomenon allows extracting rotational energy efficiently from black hole. It has been proposed by Press and Teukolsky [5] to built *black-hole bomb*. The essential of black-hole bomb mechanism is to add a reflecting mirror outside the rotating black hole. Then superradiant modes will bounce back and forth between event horizon and mirror. Meanwhile, the rotational energy extracted from black hole by means of superradiance process will grow exponentially. This mechanism has been recently restudied by many authors [6–9].

When the reflecting mirror is not artificial, superradiance amplification of impinging wave can lead to instability of black hole, which is just called superradiant instability. This kind of instability has been studied in a great number of work in recent years. For example, Kerr and Kerr-Newman black holes [10–13] and Kerr-Newman black hole immersed in magnetic field [14] are all unstable against massive scalar field perturbations, where mass terms of perturbations play the role of reflective mirrors. Five-dimensional boosted Kerr black string [15] is also unstable against massless scalar field where Kaluza-Klein momentum works as reflective mirror.

For the rotating black holes in AdS space, the boundary at infinity can also work as reflecting mirror. Small Kerr-AdS black hole in four dimensions is unstable against massless scalar field [16] and gravitational field [17] perturbations. Contrary to four-dimensional case, superradiance instability of five-dimensional rotating charged AdS black hole [18] occurs only when the orbital quantum number is even. More recently, superradiant instability of small Reissner-Nordström-anti-de Sitter black hole is investigated analytically and numerically [19]. In fact, besides AdS space, there are also other cases where the boundary at asymptotic infinity provides the reflecting mirror. For example, the rotating linear dilaton black hole [20] and the charged Myers-Perry black hole in Gödel universe [21] are unstable due to superradiance. The reason of superradiant instability for these black holes originates from the Dirichlet boundary condition at asymptotic infinity of perturbation fields.

As mentioned above, superradiant instability of charged rotating asymptotically Gödel black hole has been found by using the numerical methods [21]. In this paper, we will *re-investigate* the same aspect of this kind of rotating black holes in Gödel universe by using the analytical methods. We focus on the rotating asymptotically Gödel black hole in five-dimensional minimal supergravity theory. This black hole is also called as Kerr-Gödel black hole in literature. Firstly, by considering the scalar field perturbation in this background, we find that the asymptotically Gödel spacetime requires the wave equation to satisfy Dirichlet boundary condition at asymptotic infinity. Then, we divide the space outside the event horizon of Kerr-Gödel black hole into the near-region and the far-region and employ the matched asymptotic expansion method to solve the wave equation of scalar field perturbation. We only deal with black hole in the limit of small rotating parameter j of Gödel universe. The analysis of complex quasinormal modes by imposing the boundary conditions shows that the complex parts are positive in the regime of superradiance, which implies the growing instability of these modes. This is to say that the five-dimensional Kerr-Gödel black hole is unstable against scalar field perturbation. The reason for this instability is just the Dirichlet boundary condition at asymptotic infinity, which is similar to that of the rotating black holes in AdS space.

The remaining of this paper is arranged as follows. In Section 2, we give a brief review of Kerr-Gödel black hole in five-dimensional minimal supergravity theory. In Section 3, we investigate the classical superradiance phenomenon and the boundary condition of scalar field perturbation. In Section 4, the approximated solution of wave equation for scalar field is obtained by using the matched asymptotic expansion method and the superradiant instability is explicitly shown. The last section is devoted to conclusion and discussion.

II. FIVE-DIMENSIONAL KERR-GÖDEL BLACK HOLE

The bosonic part of five-dimensional minimal supergravity theory consists of the metric and a one-form gauge field, which are governed by Einstein-Maxwell-Chern-Simons (EMCS) equations of motion

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 2 \left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right), \\ D_{\nu} \left(F^{\mu\nu} + \frac{1}{\sqrt{3}\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta\gamma}A_{\alpha}F_{\beta\gamma} \right) &= 0. \end{aligned} \tag{1}$$

The five-dimensional Kerr-Gödel black hole is a solution to the EMCS equations of motion, the metric of which takes the form [22]

$$ds^2 = -f(r)dt^2 - q(r)r\sigma_L^3 dt - h(r)r^2(\sigma_L^3)^2 + \frac{dr^2}{v(r)} + \frac{r^2}{4}(d\theta^2 + d\psi^2 + d\phi^2 + 2\cos\theta d\psi d\phi), \quad (2)$$

where $\sigma_L^3 = d\phi + \cos\theta d\psi$, and the metric functions are given by

$$\begin{aligned} f(r) &= 1 - \frac{2M}{r^2}, \\ q(r) &= 2jr + \frac{2Ma}{r^3}, \\ h(r) &= j^2(r^2 + 2M) - \frac{Ma^2}{2r^4}, \\ v(r) &= 1 - \frac{2M}{r^2} + \frac{8jM(a + 2jM)}{r^2} + \frac{2Ma^2}{r^4}. \end{aligned} \quad (3)$$

The parameters M and a are related to the mass, and the angular momentum of black hole. In this metric, the parameter j defines the scale of the Gödel background and is responsible for the rotation of the Gödel universe [23]. When $a = 0$, this solution reduced to the Gimon-Hashimoto solution, i.e. the Schwarzschild black hole in Gödel universe [22]. The thermodynamics of this black hole has been studied in [24, 25]. The scalar field perturbation and greybody factor of Hawking radiation of this kind of black holes are also calculated in the limit of small j in [26]. This black hole has also been generalized to being charged [27] and other forms.

In this paper, we consider the non-extremal black hole case. The metric function $v(r)$ has two positive real roots r_{\pm} , which are given by

$$r_{\pm}^2 = M - 4jMa - 8j^2M^2 \pm \sqrt{\xi}, \quad (4)$$

where

$$\xi = (M - 4jMa - 8j^2M^2)^2 - 2Ma^2. \quad (5)$$

Clearly the non-extremal condition is given by $\xi > 0$. The event horizon locates at the largest root r_+ of function $v(r)$.

For latter convenience, we also present the expression of angular momentum at the event

horizon

$$\begin{aligned}\Omega_H &= \frac{2q(r_+)}{r_+(1-4h(r_+))} \\ &= \frac{4(Ma + jr_+^4)}{r_+^4 - 4j^2r_+^4(r_+^2 + 2M) + 2Ma^2} .\end{aligned}\quad (6)$$

By employing the relation of metric functions

$$q^2(r) + f(r)(1 - 4h(r)) = v(r) , \quad (7)$$

and noting that $v(r)$ vanishes at the horizon, another simple expression for the angular momentum can be derived

$$\Omega_H = \frac{2M - r_+^2}{Ma + jr_+^4} . \quad (8)$$

In the present paper, we consider the rotating parameters a and j are both positive. From the expression of r_+^2 in (4), one can easily find $r_+^2 < 2M$, which implies that the angular momentum Ω_H is always positive when the rotating parameters a and j are positive.

III. SUPERRADIANCE AND BOUNDARY CONDITION

Now let us consider the wave equation of massless scalar field perturbation in the background (2), which is given by Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \Phi = \frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi) = 0 . \quad (9)$$

Because the metric has the killing vectors ∂_t , ∂_ψ , and ∂_ϕ , we can take the ansatz of scalar field as

$$\Phi = e^{-i\omega t + i n \psi + i m \phi} \Theta(\theta) R(r) . \quad (10)$$

Substituting this ansatz into the wave equation (9) and separating the variables, we can get the angular equation

$$\frac{1}{\sin \theta} \partial_\theta [\sin \theta \partial_\theta \Theta(\theta)] - \frac{(n - m \cos \theta)^2}{\sin^2 \theta} \Theta(\theta) + [l(l+1) - m^2] \Theta(\theta) = 0 , \quad (11)$$

and the radial equation

$$\begin{aligned}\frac{1}{4r} \partial_r [r^3 v(r) \partial_r R(r)] + \frac{r^2(1-4h(r))}{4v(r)} \left[\omega - \frac{2mq(r)}{r(1-4h(r))} \right]^2 R(r) \\ + \left[l(l+1) + \frac{4m^2 h(r)}{(1-4h(r))} \right] R(r) = 0 .\end{aligned}\quad (12)$$

Obviously, the angular equation (11) is independent of the black hole parameters and is exactly solvable. The solutions for the angular equation are just the spin-weighted spherical harmonics functions, where the integers $l = 0, 1, 2, \dots$ are the separation constants and the modes $m = 0, \pm 1, \dots, \pm l$.

In the following, we will specify the appropriate boundary conditions for the instability problem. At the horizon, the third set of terms in radial wave equation (12) can be neglected and this equation can be reduced to the form

$$v\partial_r(v\partial_r R(r)) + (1 - 4h(r_+))(\omega - m\Omega_H)^2 R(r) = 0. \quad (13)$$

Near the horizon, we use the approximation $v(r) \cong 2(r_+^2 - r_-^2)(r - r_+)/r_+^3$. Then the solution of equation (13) satisfying the ingoing boundary condition at the horizon is given by

$$R(r) \sim (r - r_+)^{-i\varpi} = e^{-i\varpi \ln(r - r_+)}, \quad (14)$$

where we have defined

$$\varpi = \frac{r_+(r_+^4 + 2Ma^2 - 4j^2 r_+^6 - 8j^2 M r_+^2)^{1/2}}{2(r_+^2 - r_-^2)}(\omega - m\Omega_H). \quad (15)$$

This solution gives us the superradiance condition of five-dimensional Kerr-Gödel black hole. When the frequency of the wave is such that ϖ is negative, i.e. $\omega < m\Omega_H$, one is in the superradiant regime, and the amplitude of an ingoing bosonic field is amplified after scattering by the event horizon. Meanwhile, for the present purpose, it is enough to consider the frequency ω is positive, which gives the superradiance condition as

$$0 < \omega < m\Omega_H. \quad (16)$$

From this condition, one can see that superradiance will occur only for the positive m . In the following, we will only work with the positive m .

At infinity, the radial wave equation (12) is dominated by

$$r\partial_r^2 R(r) + 3\partial_r R(r) - 4j^2\omega^2 r^3 R(r) = 0. \quad (17)$$

The solution is given by

$$R(r) \sim \frac{1}{r^2} e^{-j\omega r^2}, \quad (18)$$

where we have used the analogy with AdS backgrounds and imposed the Dirichlet boundary conditions at spatial infinity.

With the boundary conditions that the ingoing wave at the horizon and the Dirichlet boundary condition at the infinity, one can solve the complex quasinormal modes of the massless scalar field in Kerr-Gödel background. If the imaginary part of quasinormal mode is negative, it is known that the system is stable against this kind of perturbation. The instability means that the imaginary part is positive. In the next section, we will calculate the quasinormal modes by using the matching technique. It is shown that, in the regime of superradiance, the imaginary part of quasinormal mode is positive. In other words, the superradiant instability of five-dimensional Kerr-Gödel black hole can be found analytically.

IV. ANALYTICAL CALCULATION OF SUPERRADIANT INSTABILITY

In this section, we will present an analytical calculation of superradiant instability for the massless scalar perturbation. We will adopt the so-called matched asymptotic expansion method to solve the radial wave equation (12). It turns out to be convenient to use the new variable x defined by $x = r^2$. Then the radial wave equation can be transformed into

$$\begin{aligned} \Delta \partial_x (\Delta \partial_x) R(x) + \frac{x^3}{4} (1 - 4h(x)) \left[\omega - \frac{2mq(x)}{\sqrt{x}(1 - 4h(x))} \right]^2 R(x) \\ + \Delta \left[l(l+1) + \frac{4m^2 h(x)}{1 - 4h(x)} \right] R(x) = 0, \end{aligned} \quad (19)$$

where we have used $\Delta = x^2 v(x) = (x - x_+)(x - x_-)$ with $x_{\pm} = r_{\pm}^2$.

In order to employ the matched asymptotic expansion method, we should take the assumption $\omega M \ll 1$, and divide the space outside the event horizon into two regions, namely, a near-region, $x - x_+ \ll 1/\omega$, and a far-region, $x - x_+ \gg M$. The approximated solution can be obtained by matching the near-region solution and the far-region solution in the overlapping region $M \ll x - x_+ \ll 1/\omega$.

Previous numerical works [21, 28] on the spectrum of asymptotically Gödel black holes show a number of common features with the spectrum of AdS spacetime, where the rotational parameter j of Gödel universe plays the role of the inverse AdS radius ℓ . Inspired by the work of [16], where small AdS black hole are considered, we will deal with the rotating asymptotically Gödel black hole in the limit of small rotational parameter j in the following. The small AdS black hole condition implies that $r_+/\ell \ll 1$. For the small Gödel black hole, we assume that $jr_+ \ll 1$.

With these assumptions, we can analyse the properties of the solution and study the stability of black hole against the perturbation by imposing the appropriate boundary conditions obtained in the last section.

A. Near-region solution

Firstly, Let us focus on the near-region in the vicinity of the event horizon, $\omega(x - x_+) \ll 1$. For the small j black holes, this means $jr_+ \ll 1$. The radial wave function (19) in the near-region can be reduced to the form

$$\Delta \partial_x (\Delta \partial_x R(x)) + [(x_+ - x_-)^2 \varpi^2 - l(l+1)\Delta] R(x) = 0 . \quad (20)$$

Noted that the last term in Eq.(19) is neglected because we only consider the case $m \sim \omega$.

Introducing the new coordinate variable

$$z = \frac{x - x_+}{x - x_-} , \quad (21)$$

the near-region radial equation can be written in the form of

$$z \partial_z (z \partial_z R(z)) + \left[\varpi^2 - l(l+1) \frac{z}{(1-z)^2} \right] R(z) = 0 , \quad (22)$$

with

$$\varpi = \frac{r_+(r_+^4 + 2Ma^2)^{1/2}}{2(r_+^2 - r_-^2)} (\omega - m\Omega_H) . \quad (23)$$

This expression for ϖ is coincide with the expression given in (15) in the small j limit.

Through defining

$$R = z^{i\varpi} (1-z)^{l+1} F(z) , \quad (24)$$

the near-region radial wave equation becomes

$$z(1-z) \partial_z^2 F(z) + [c - (1+a+b)] \partial_z F(z) - abF(z) = 0 , \quad (25)$$

with the parameters

$$\begin{aligned} a &= l+1 + 2i\varpi , \\ b &= l+1 , \\ c &= 1 + 2i\varpi . \end{aligned} \quad (26)$$

In the neighborhood of $z = 0$, the general solution of the radial wave equation is given in terms of the hypergeometric function

$$R = Az^{-i\varpi}(1-z)^{l+1}F(l+1, l+1-2i\varpi, 1-2i\varpi, z) \\ + Bz^{i\varpi}(1-z)^{l+1}F(l+1, l+1+2i\varpi, 1+2i\varpi, z). \quad (27)$$

It is obvious that the first term represents the ingoing wave at the horizon, while the second term represents the outgoing wave at the horizon. Because we are considering the classical superradiance process, the ingoing boundary condition at the horizon should be employed. Then we have to set $B = 0$. The physical solution of the radial wave equation corresponding to the ingoing wave at the horizon is then given by

$$R = Az^{-i\varpi}(1-z)^{l+1}F(l+1, l+1-2i\varpi, 1-2i\varpi, z). \quad (28)$$

In order to match the far-region solution that will be obtained in the next subsection, we should study the large r , $z \rightarrow 1$, behavior of the near-region solution. For the sake of this purpose, we can use the $z \rightarrow 1-z$ transformation law for the hypergeometric function

$$F(a, b, c, z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, a+b-c+1, 1-z) \\ + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ \times F(c-a, c-b, c-a-b+1, 1-z). \quad (29)$$

By employing this formula and using the properties of hypergeometric function $F(a, b, c, 0) = 1$, we can get the large r behavior of the near-region solution as

$$R \sim A\Gamma(1-2i\varpi) \left[\frac{(r_+^2 - r_-^2)^{-l}\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-2i\varpi)} r^{2l} \right. \\ \left. + \frac{(r_+^2 - r_-^2)^{l+1}\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l-2i\varpi)} r^{-2l-2} \right], \quad (30)$$

where the variable x has been restored to r for later convenience. This solution should be matched with the small r behavior of the far-region solution obtained in the next subsection.

B. Far-region solution

In the Far-region, $x - x_+ \gg M$, we can neglect the effects induced by the black hole, i.e. we have $a \sim 0$ and $M \sim 0$. The metric functions can be approximated as $v(x) = f(x) = 1$,

$h(x) = j^2 x$, and $q(x) = 2j\sqrt{x}$. One can deduce the far-region radial wave equation as

$$\partial_x^2(xR) + \left[-j^2\omega^2 + \frac{\omega(\omega - 8mj)}{4x} - \frac{l(l+1)}{x^2} \right] (xR) = 0. \quad (31)$$

By defining the new variable $\zeta = 2j\omega x$, the far-region radial wave equation can be reduced to

$$\partial_\zeta^2(\zeta R) + \left[-\frac{1}{4} + \frac{\rho}{\zeta} - \frac{l(l+1)}{\zeta^2} \right] (\zeta R) = 0, \quad (32)$$

with the parameter $\rho = (\omega - 8mj)/8j$.

This is a standard Whittaker equation $\partial_\zeta^2 W + [-1/4 + \rho/\zeta + (1/4 - \mu^2)/\zeta^2]W = 0$ with $W = \zeta R$ and $\mu = l + 1/2$. The general solution is given by $W = \zeta^{\mu+1/2} e^{-\zeta/2} [\alpha M(\tilde{a}, \tilde{b}, \zeta) + \beta U(\tilde{a}, \tilde{b}, \zeta)]$, where M and U are Whittaker's functions with $\tilde{a} = 1/2 + \mu - \rho$ and $\tilde{b} = 1 + 2\mu$. So the far-region solution of the radial wave equation is given by

$$R = \zeta^l e^{-\zeta/2} [\alpha M(l+1-\rho, 2l+2, \zeta) + \beta U(l+1-\rho, 2l+2, \zeta)]. \quad (33)$$

Now we want to impose the boundary condition at asymptotic infinity. We are interested in the superradiance region with $0 < \omega < m\Omega_H$, so we have $\zeta = 2j\omega r^2 \rightarrow +\infty$ when $r \rightarrow +\infty$. When $\zeta \rightarrow +\infty$, by using the properties of the Whittaker's functions $M(\tilde{a}, \tilde{b}, \zeta) \sim \zeta^{\tilde{a}-\tilde{b}} e^{\zeta} \Gamma(\tilde{b})/\Gamma(\tilde{a})$ and $U(\tilde{a}, \tilde{b}, \zeta) \sim \zeta^{-\tilde{a}}$, one can get the large r behavior of the far-region solution as

$$R \sim \alpha \frac{\Gamma(2l+2)}{\Gamma(l+1-\rho)} (2j\omega r^2)^{-1-\rho} e^{j\omega r^2} + \beta (2j\omega r^2)^{-1-\rho} e^{-j\omega r^2}. \quad (34)$$

Obviously the first term is divergent at asymptotic infinity. To match the Dirichlet boundary condition at infinity, we have to set $\alpha = 0$. Thus the far-region solution with the Dirichlet boundary condition at asymptotic infinity is given by

$$R = \beta (2j\omega)^l r^{2l} e^{-j\omega r^2} U(l+1-\rho, 2l+2, 2j\omega r^2). \quad (35)$$

This solution is just the solution of scalar field wave equation in the background of the pure five-dimensional Gödel spacetime [28–30].

We assume for a moment that we have no black hole, and calculate the real frequencies that can propagate in the pure five-dimensional Gödel spacetime. In this setup, the spacetime geometry is horizon-free, and the solution of the scalar field perturbation in the background of the pure five-dimensional Gödel spacetime should be regular at the origin $r = 0$.

When $\zeta \rightarrow 0$, using the properties of Whittaker's function $U(\tilde{a}, \tilde{b}, \zeta) \sim \zeta^{1-\tilde{b}}\Gamma(\tilde{b}-1)/\Gamma(\tilde{a})$, one can get the small r behavior of the far-region solution as

$$R \sim \beta(2j\omega)^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(l+1-\rho)} r^{-2l-2}. \quad (36)$$

So, when $r \rightarrow 0$, $r^{-2l-2} \rightarrow \infty$, and the solution diverges. To have a regular solution at the origin $r = 0$, we must demand that $\Gamma(l+1-\rho) \rightarrow \infty$. This occurs when the argument of the gamma function is a non-positive integer. Therefore, we have the condition

$$l+1-\rho = -N, \quad N = 0, 1, 2, \dots. \quad (37)$$

So the requirement of the regularity of the wave solution at the origin selects the frequencies of the scalar field that might propagate in the pure five-dimensional Gödel spacetime

$$\omega_N = 8j(N+l+m+1). \quad (38)$$

Now let us come back to the Kerr-Gödel black hole case. In the spirit of [16], we expect that there will be a small imaginary part δ in the allowed frequencies induced by the black hole event horizon

$$\omega = \omega_N + i\delta. \quad (39)$$

From $\Psi \sim e^{-i\omega t}$, one can see that the small imaginary δ describes the slow growing instability of the modes when $\delta > 0$. Our task is to prove that δ is positive in the regime of superradiance.

Inserting this expression for the frequency ω , one can get the far-region solution of the radial wave equation as

$$R = \beta(2j\omega)^l r^{2l} e^{-j\omega r^2} U(-N - i\delta/8j, 2l+2, 2j\omega r^2). \quad (40)$$

In order to match the far-region solution with the near-region solution, we need to find the small r behavior of the far-region solution. It is known that the Whittaker's function $U(\tilde{a}, \tilde{b}, \zeta)$ can be expressed in terms of the Whittaker's function $M(\tilde{a}, \tilde{b}, \zeta)$. By inserting this relation on the far-region solution (40), we can show that the far-region solution can be rewritten as

$$R = \beta(2j\omega)^l r^{2l} e^{-j\omega r^2} \frac{\pi}{\sin \pi(2l+2)} \left[\frac{M(-N - i\delta/8j, 2l+2, 2j\omega r^2)}{\Gamma(-N - 2l - 1 - i\delta/8j)\Gamma(2l+2)} - (2j\omega)^{-2l-1} r^{-4l-2} \frac{M(-N - 2l - 1 - i\delta/8j, -2l, 2j\omega r^2)}{\Gamma(-N - i\delta/8j)\Gamma(-2l)} \right]. \quad (41)$$

Applying to this expression the functional expressions for the gamma functions

$$\begin{aligned}\Gamma(n+1) &= n! , \\ \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin \pi z} ,\end{aligned}\tag{42}$$

it is easy to show that

$$\frac{1}{\Gamma(-2l)} = -\frac{\sin \pi(2l+2)}{\pi}(2l)! ,\tag{43}$$

and for the small δ

$$\begin{aligned}\frac{1}{\Gamma(-N-2l-1-i\delta/8j)} &= (-1)^N \frac{\sin \pi(2l+2)}{\pi}(N+2l+1)! , \\ \frac{1}{\Gamma(-N-i\delta/8j)} &= (-1)^{N+1} N! i\delta/8j .\end{aligned}\tag{44}$$

Then by using the property of Whittaker's function $M(\tilde{a}, \tilde{b}, 0) = 1$, one can get the small r behavior of the far-region solution as

$$R = \beta(-1)^N (2j\omega_N)^l \left[\frac{(N+2l+1)!}{(2l+1)!} r^{2l} - i\delta \frac{(2l)!N!}{2^{2l+4} j^{2l+2} \omega_N^{2l+1}} r^{-2l-2} \right] .\tag{45}$$

C. Matching condition: the unstable modes

By comparing the large r behavior of the near-region solution with the small r behavior of the far-region solution, one can conclude that there exists the overlapping region $M \ll x - x_+ \ll 1/\omega$ where the two solutions should match. In this region, the matching of the near-region solution in the large r region (30) and the far-region solution in the small r region (45) yields the allowed values of the small imaginary part δ in the frequency ω

$$\delta \cong -\sigma(\omega_N - m\Omega_H) r_+ (r_+^4 + 2Ma^2)^{1/2} (r_+ - r_-)^{2l} j^{2l+2} ,\tag{46}$$

where

$$\sigma = 2^{2l+4} \omega_N^{2l+1} \frac{(l!)^2 (2l+1+N)!}{((2l)!(2l+1)!)^2 N!} \left[\prod_{k=1}^l (k^2 + 4\varpi^2) \right] ,\tag{47}$$

with $\varpi = (\omega_N - m\Omega_H) r_+ (r_+^4 + 2Ma^2)^{1/2} / 2(r_+^2 - r_-^2)$. So, we have

$$\delta \propto -(Re[\omega] - m\Omega_H) .\tag{48}$$

It is easy to see that, in the superradiance regime, $Re[\omega] - m\Omega_H < 0$, the imaginary part of the complex frequency $\delta > 0$. The scalar field has the time dependence $e^{-i\omega t} = e^{-i\omega_N t} e^{\delta t}$, which implies the exponential amplification of superradiance modes. This will lead to the instability of these modes.

From the normal modes in pure five-dimensional Gödel spacetime (38), we can see that $Re[\omega] \sim j$. We have assumed that $\omega M \ll 1$. So we have $jM \ll 1$, which is consistent with the small Gödel black hole assumption $jr_+ \ll 1$ because $M \sim r_+^2$. This is to say that the two assumptions we have made in this section are consistent with each other.

At last, we can conclude that the five-dimensional small Kerr-Gödel black hole is unstable against the massless scalar field perturbation. This instability is caused by the superradiance of the scalar field.

V. CONCLUSION

This paper is devoted to an analytical study of superradiant instability of five-dimensional small Kerr-Gödel black hole. This instability has been found by R. A. Konoplya and A. Zhidenko using the numerical methods previously in [21]. Generally, superradiant instability naturally happens when two conditions are satisfied: (1) Black hole has rotation; (2) There is a natural reflecting mirror outside the black hole. In the present case, Dirichlet boundary condition at infinity for the asymptotically Gödel black hole, which is obtained in section 3 by analogy with the AdS background, plays the role of reflecting mirror. We have adopted the analytical methods which is used in [16] to study the superradiant instability of small Kerr-AdS black hole. We assume that the energy of scalar field perturbation is low and the scale of Gödel black hole is small. Then we divide the space outside the event horizon into the near-region and the far-region and employ the matched asymptotic expansion method to solve the scalar field wave equation. The analysis of complex quasinormal modes explicitly shows that the complex parts are positive in the regime of superradiance, which implies the growing instability of these modes. This is to say that the five-dimensional small Kerr-Gödel black hole is unstable against scalar field perturbation.

As is well-known, the gravitational perturbation will also undergo an superradiant amplification when scattered by the event horizon. Unlike the simply rotating Kerr-AdS black hole, where the superradiant instability of a massless scalar field implies the gravitational

instability [31], we can not obtain the similar conclusion for asymptotically Gödel black hole directly. So it will be interesting to study the gravitational (in)stability of five-dimensional Kerr-Gödel black hole. Because of the complexity of the metric and the field equation, it will be a challenging project for future work.

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